MATH20132 Calculus of Several Variables. 2019-20

Solutions to Problems 6: Graphs, level sets, parametric sets, Implicit & Inverse functions

Surfaces as a level set.

1. Are the following level sets surfaces? (Look at the Jacobian matrices of the level sets).

i. $\{\mathbf{x} \in \mathbb{R}^3 : x^2 + 3y^2 + 2z^2 = 9\}$,

ii. The set of $\mathbf{x} \in \mathbb{R}^3$ satisfying

$$\begin{aligned} x^2 + y^2 - z^2 &= 1, \\ x^2 + 3y^2 + 2z^2 &= 9. \end{aligned}$$

iii. The set of $\mathbf{x} \in \mathbb{R}^3$ satisfying

$$\begin{aligned} x^2 + y^2 - z^2 &= 11, \\ x^2 + 3y^2 + 2z^2 &= 9. \end{aligned}$$

iv. The set of $\mathbf{x} \in \mathbb{R}^4$ satisfying

$$\begin{aligned} 3x + 2y^2 + u^2 + v^2 &= 13, \\ x^3 - y^3 + u^3 - v^3 &= 0, \\ 3x^3 + 5y + 5u^2 - v^2 &= 24. \end{aligned}$$

Hint the point $\mathbf{p} = (1, 1, 2, 2)^T$ may be of interest.

Solution i. With $f(\mathbf{x}) = x^2 + 3y^2 + 2z^2 - 9$ the Jacobian matrix is $Jf(\mathbf{x}) = (2x, 6y, 4z)$. Is this full-rank for $\mathbf{x} : f(\mathbf{x}) = 0$?

To talk of one vector being linear independent is that you cannot find a non-zero coefficient which you multiply your vector to get the zero vector. You can only do this if you start with the zero vector.

Is our Jacobian matrix ever $\mathbf{0}$? Yes, when $\mathbf{x} = \mathbf{0}$. But $f(\mathbf{0}) = -9 \neq 0$ so $\mathbf{0}$ is not in the level set. Thus for all points in the level set the Jacobian matrix is non-zero and so the level set **is** a surface.

ii. With

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x^2 + y^2 - z^2 - 1 \\ x^2 + 3y^2 + 2z^2 - 9 \end{pmatrix},$$

the Jacobian matrix is

$$J\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 2x & 2y & -2z \\ 2x & 6y & 4z \end{pmatrix}.$$

Is this full-rank, i.e. the two rows linearly independent, for $\mathbf{x} : \mathbf{f}(\mathbf{x}) = \mathbf{0}$? Remember, two vectors are linearly *dependent* iff one is a scalar multiple of the other. Thus $J\mathbf{f}(\mathbf{x})$ will **not** be of full rank for $\mathbf{x}$ if there exists $\lambda \in \mathbb{R}$ such that

$$\begin{pmatrix} 2x & 6y & 4z \end{pmatrix} = \lambda \begin{pmatrix} 2x & 2y & -2z \end{pmatrix}.$$

That is, $x = \lambda x$, $3y = \lambda y$ and $2z = -\lambda z$.

From $x = \lambda x$ either $x = 0$ or $\lambda = 1$.

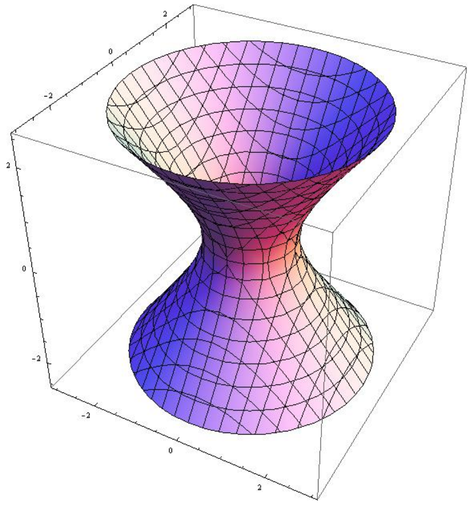
- If $\lambda = 1$ then the last two equations become $y = 3y$ and $z = -2z$, i.e. $y = z = 0$. So the two rows are linearly dependent if $y = z = 0$. Yet, for $\mathbf{x} : y = z = 0$ we have $\mathbf{f}(\mathbf{x}) = (x^2 - 1, x^2 - 9)^T \neq \mathbf{0}$ for all $x \in \mathbb{R}$.

- If $x = 0$ then look at $3y = \lambda y$. For this, either $\lambda = 3$ or $y = 0$.

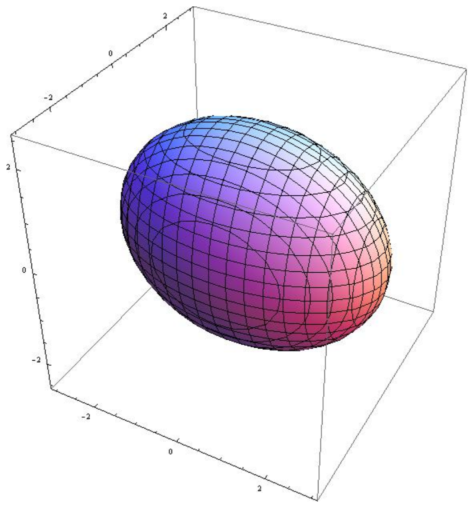
  * If $x = 0$ and $\lambda = 3$ then, from $2z = -\lambda z = -3z$ we get $z = 0$. So the two rows are linearly dependent if $x = z = 0$. Yet, for $\mathbf{x} : x = z = 0$ we have $\mathbf{f}(\mathbf{x}) = (y^2 - 1, 3y^2 - 9)^T \neq \mathbf{0}$ for all $y \in \mathbb{R}$.
  * If $x = 0$ and $y = 0$ we can see the two rows are linearly dependent (with $\lambda = -2$). Yet, for $\mathbf{x} : x = y = 0$ we have $\mathbf{f}(\mathbf{x}) = (-z^2 - 1, 2z^2 - 9)^T \neq \mathbf{0}$ for all $z \in \mathbb{R}$.

We have found many points at which $J\mathbf{f}(\mathbf{x})$ is not of full-rank but none satisfy $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. Hence the level set **is** a surface.

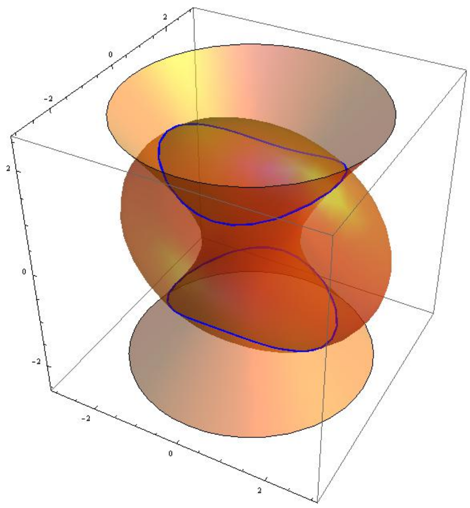
The set of points $x^2 + y^2 - z^2 = 1$ is an Hyperboloid:



The set of points $x^2 + 3y^2 + 2z^2 = 9$ is an ellipsoid:



Their intersection is a disjoint union of two closed lines in $\mathbb{R}^3$, here shown in blue.



iii. Be Careful! You can go through the argument of part ii but in this case the level set is **empty**! Subtract equation 1 from 2 to find that any points on the level set must satisfy $2y^2 + 2z^2 = -2$, impossible.

iv. With

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 3x + 2y^2 + u^2 + v^2 - 13 \\ x^3 - y^3 + u^3 - v^3 \\ 3x^3 + 5y + 5u^2 - v^2 - 24 \end{pmatrix}$$

the Jacobian matrix is

$$J\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 3 & 4y^2 & 2u & 2v \\ 3x^2 & -3y^2 & 3u^2 & -3v^2 \\ 9x^2 & 5 & 10u & -2v \end{pmatrix}.$$

The point $\mathbf{p} = (1, 1, 2, 2)^T$ is of interest because $\mathbf{f}(\mathbf{p}) = \mathbf{0}$, so $\mathbf{p}$ is in the level set. But also

$$J\mathbf{f}(\mathbf{p}) = \begin{pmatrix} 3 & 4 & 4 & 4 \\ 3 & -3 & 12 & -12 \\ 9 & 5 & 20 & -4 \end{pmatrix}.$$

This is **not** of full rank because $2r_1 + r_2 = r_3$. Hence the level set is **not** a surface at $\mathbf{p}$.

Surfaces as an image set.

2. Are the following parametrically defined sets surfaces? Give your reasons. (Look at their Jacobian matrices.)

i. $\left\{ \left(x^2+y^2,\, xy,\, 2x-3y\right)^T : x, y \in \mathbb{R} \right\},$

ii. $\left\{ \left(x^2+y^2,\; xy,\, 2x^3-3y^2\right)^T : (x,y) \in \mathbb{R}^2 \backslash \{\mathbf{0}\} \right\},$

iii. $\left\{ \left(x^2+y^2,\, xy,\, 2x^3-3y^2\right)^T : x>0, y>0 \right\},$

iv. $\left\{ \left(ye^x,\, xe^y,\, 1\right)^T : x, y \in \mathbb{R} \right\}.$

Solution For the image set to be a surface the Jacobian $J\mathbf{F}(\mathbf{x})$ has to be of full-rank at all points.

i. Here the set given is the image set of the function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^3$,

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} x^2+y^2 \\ xy \\ 2x-3y \end{pmatrix},$$

which has the Jacobian matrix

$$J\mathbf{F}(\mathbf{x}) = \begin{pmatrix} 2x & 2y \\ y & x \\ 2 & -3 \end{pmatrix}.$$

A quick observation shows that when $\mathbf{x} = \mathbf{0}$ the columns $(0,0,2)^T$ and $(0,0,-3)^T$ are not linearly independent. Hence $\operatorname{Im}\mathbf{F} = \{\mathbf{F}(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^2\}$ is **not** a surface.

ii. The Jacobian matrix is

$$J\mathbf{F}(\mathbf{x}) = \begin{pmatrix} 2x & 2y \\ y & x \\ 6x^2 & -6y \end{pmatrix}.$$

The columns are obviously linearly dependent if $\mathbf{x} = \mathbf{0}$ but that point has been omitted. We look, instead, for $\mathbf{x} \neq \mathbf{0}$ for which there exists $\lambda \in \mathbb{R}$ such that

$$\begin{pmatrix} 2x \\ y \\ 6x^2 \end{pmatrix} = \lambda \begin{pmatrix} 2y \\ x \\ -6y \end{pmatrix}.$$

That is, $x = \lambda y$, $y = \lambda x$ and $x^2 = -\lambda y$. The first two combine as $x = \lambda^2 x$ in which case either $x = 0$ or $\lambda^2 = 1$.

- If $x = 0$ then $y = \lambda x = 0$ but $(0, 0)^T$ has been omitted.
- This leaves $\lambda^2 = 1$, i.e. either $\lambda = 1$ or $\lambda = -1$.
  - ∗ If $\lambda = 1$ then $x = y$ and $x^2 = -y = -x$. Since $x \neq 0$ we have $x = -1$ and thus $y = -1$. That is, at $(-1, 1)^T$ the Jacobian is not of full rank. Hence the given set is **not** a surface.
  - ∗ If $\lambda = -1$ then $x = -y$ and $x^2 = y = -x$. This implies $x = -1$ and $y = 1$. So $(-1, 1)^T$ is another (and the last) point at which the Jacobian matrix is not of full rank.

iii. We have the same Jacobian matrix as in part ii. We saw there that if, and only if, $(x, y)^T = \mathbf{0}$, $(-1, 1)^T$ or $(-1, -1)^T$ then $J\mathbf{F}(\mathbf{x})$ is not of full-rank. These three points do not lie in the region $x > 0, y > 0$ and so $\{\mathbf{F}(\mathbf{x}) : x > 0, y > 0\}$ **is** a surface.

iv. In this example we have

$$J\mathbf{F}(\mathbf{x}) = \begin{pmatrix} ye^x & e^x \\ e^y & xe^y \\ 0 & 0 \end{pmatrix}.$$

Does there exist $\mathbf{x} = (x, y)^T$ and λ such that $ye^x = \lambda e^x$ and $e^y = \lambda xe^y$? Since $e^x, e^y \neq 0$ divide to get $y = \lambda$ and $1 = \lambda x$. Substitute in to get that the parametric set **fails** to be a surface at all points on the two hyperbola given by $xy = 1$.

Graphs in $\mathbb{R}^3$.

3. Suppose that $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is Fréchet differentiable on U. Let

$$G_f = \left\{ \begin{pmatrix} \mathbf{a} \\ f(\mathbf{a}) \end{pmatrix} : \mathbf{a} \in U \right\} \subseteq \mathbb{R}^3.$$

be the graph of f.

Prove that as $\mathbf{a} \in U$ varies in the $\mathbf{v} \in \mathbb{R}^2$ direction the directional derivative $d_{\mathbf{v}} f(\mathbf{a})$ represents the rate of change in the z-coordinate of the corresponding points on the graph.

Hint look at the rate of change of going from

$$\begin{pmatrix} \mathbf{a} \\ f(\mathbf{a}) \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} \mathbf{a} + t\mathbf{v} \\ f(\mathbf{a} + t\mathbf{v}) \end{pmatrix}$$

Solution Following the hint, the rate of change from

$$\begin{pmatrix} \mathbf{a} \\ f(\mathbf{a}) \end{pmatrix} \text{ to } \begin{pmatrix} \mathbf{a} + t\mathbf{v} \\ f(\mathbf{a} + t\mathbf{v}) \end{pmatrix}$$

is, if it exists,

$$\begin{aligned} \lim_{t \to 0+} \frac{1}{t} \begin{pmatrix} \mathbf{a} + t\mathbf{v} - \mathbf{a} \\ f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a}) \end{pmatrix} &= \begin{pmatrix} \mathbf{v} \\ \lim_{t \to 0+} (f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a}))/t \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{v} \\ d_{\mathbf{v}} f(\mathbf{a}) \end{pmatrix}. \end{aligned}$$

4. Let $f(\mathbf{x}) = 4 - 3x^2 + xy - y^2, \mathbf{x} \in \mathbb{R}^2$. If a spider stands on the graph of f above $\mathbf{q} = (1,1)^T$ in which direction should the spider move for

i. the fastest ascent?

ii. the fastest descent?

iii. to stay at the same height?

Remember, though the graph lies within $\mathbb{R}^3$ the direction will be in $\mathbb{R}^2$; we see this in real life when, on a mountain, you only give directions using West & North coordinates, no mention is given of up or down.

Hint Look back at Question 9 on Sheet 5 that looked at bounds on $d_{\mathbf{v}} f(\mathbf{a})$ and when they are attained.

Solution The gradient vector is

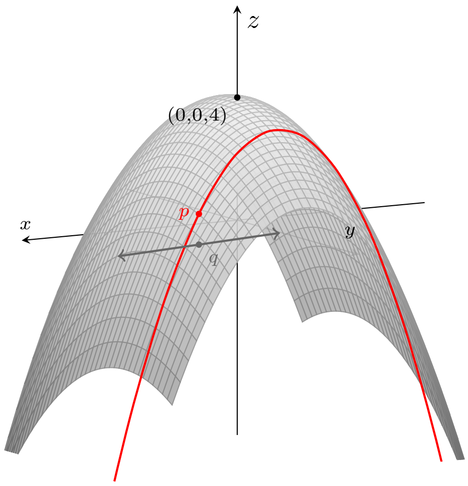
$$\nabla f(\mathbf{x}) = \begin{pmatrix} -6x + y \\ x - 2y \end{pmatrix} \quad \text{so} \quad \nabla f(\mathbf{q}) = \begin{pmatrix} -5 \\ -1 \end{pmatrix}.$$

i. The quickest ascent will be $\max_{\mathbf{v}} d_{\mathbf{v}} f(\mathbf{q})$. From the Question 9 on Sheet 5 this is in the direction of the gradient vector, i.e. $(-5,-1)^T/\sqrt{26}$ (the 'direction' should be a unit vector).

ii. The quickest descent will be $\min_{\mathbf{v}} d_{\mathbf{v}} f(\mathbf{q})$. Again from Question 9, Sheet 5, this is in the opposite direction of the gradient vector, i.e. $(5,1)^T/\sqrt{26}$.

iii. To stay at the same height we require no change in the z-coordinate, i.e. $d_{\mathbf{v}} f(\mathbf{q}) = 0$. So we need to solve $\nabla f(\mathbf{q}) \bullet \mathbf{v} = 0$, i.e. $(-5,-1)^T \bullet \mathbf{v} = 0$. Hence $\mathbf{v} = \pm (1,-5)^T/\sqrt{26}$.

The graph $z = f(\mathbf{x})$ for Question 4::



As we look at the surface the point $\mathbf{q}$ lies in the x - y plane underneath the surface. What is perhaps interesting here is that the path of greatest ascent and descent (here in red) does not go towards the highest point of the surface (at $(0, 0, 4)^T$).

5. Define the function

$$f(\mathbf{x}) = (x-1)^2 + y^2 \quad \text{for} \quad \mathbf{x} = (x, y)^T \in \mathbb{R}^2.$$

Imagine standing on the graph of f above the point $\mathbf{q} = (0, 2)^T$ and spilling water. In which direction would the water flow?

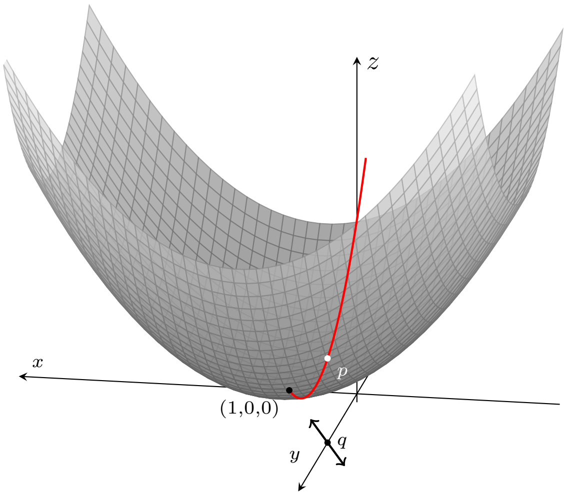
Solution Recall that $d_{\mathbf{v}} f(\mathbf{q}) = \nabla f(\mathbf{q}) \bullet \mathbf{v}$ and so $d_{\mathbf{v}} f(\mathbf{q})$ is greatest when $\mathbf{v} = \nabla f(\mathbf{q}) / |\nabla f(\mathbf{q})|$, least when $\mathbf{v} = -\nabla f(\mathbf{q}) / |\nabla f(\mathbf{q})|$, and zero when $\mathbf{v}$ is orthogonal to $\nabla f(\mathbf{q})$.

In the present example,

$$\nabla f(\mathbf{q}) = \begin{pmatrix} 2\,(x-1) \\ 2y \end{pmatrix}_{\mathbf{x}=\mathbf{q}} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}.$$

So the greatest ascent is in the direction $(-1, 2)^T / \sqrt{5}$, greatest descent in $(1, -2)^T / \sqrt{5}$ and no height change in $\pm\,(2, 1)^T / \sqrt{5}$. This is not a physics course but it is not unreasonable to assume that water will follow the path of steepest descent, i.e. go in the direction $(1, -2)^T / \sqrt{5}$.

The graph of $f(\mathbf{x}) = (x-1)^2 + y^2$ for Question 5:



Note that $(x-1)^2 + y^2 \geq 0$ with equality when $(x, y) = (1, 0)$. So the lowest point of the paraboloid is, in $\mathbb{R}^3$, at $(1, 0, 0)^T$.

The direction from $\mathbf{q} = (0, 2)^T$ to the lowest point $(1, 0)^T$ is $(1, -2)/\sqrt{5}$, and I would suggest it is no surprise that this was the direction of steepest descent, the direction the water would take.

Graphs as an image set and a level set

6. Define $\boldsymbol{\phi} : \mathbb{R}^2 \to \mathbb{R}^2, (x, y)^T \to (xy^2, x^2 + y)^T$.

i. The graph $G_{\boldsymbol{\phi}}$ is the image of some function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^4$. Find $\mathbf{F}$ and the Jacobian matrix $J\mathbf{F}(\mathbf{x})$.

ii. The graph $G_{\boldsymbol{\phi}}$ can be expressed as a level set of a system of equations. Find such a system of equations and find the Jacobian matrix of the system.

   Hint Since $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^4$ write $\mathbf{F}(\mathbf{x}) = (s, t, u, v)^T$ and find relations between the s, t, u and v.

Solution i. $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^4$ is given by

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\phi}(\mathbf{x}) \end{pmatrix}, \tag{1}$$

for $\mathbf{x} \in \mathbb{R}^2$. Equivalently,

$$\mathbf{F}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ xy^2 \\ x^2+y \end{pmatrix}.$$

From this second form we see that

$$J\mathbf{F}(\mathbf{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ y^2 & 2xy \\ 2x & 1 \end{pmatrix} = \begin{pmatrix} I_2 \\ J\boldsymbol{\phi}(\mathbf{x}) \end{pmatrix},$$

where I_2 is the 2×2 identity matrix. Hopefully you could have derived this last form for $J\mathbf{F}(\mathbf{x})$ directly from (1).

ii. Following the hint write

$$\mathbf{F}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ xy^2 \\ x^2+y \end{pmatrix} = \begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix}.$$

We find that $u = xy^2 = st^2$ and $v = x^2 + y = s^2 + t$. Thus the level set is

$$\begin{aligned} u - st^2 &= 0 \\ v - s^2 - t &= 0. \end{aligned}$$

The Jacobian Matrix is

$$\begin{pmatrix} -t^2 & -2st & 1 & 0 \\ -2s & -1 & 0 & 1 \end{pmatrix}. \tag{2}$$

Note you may have written the system as

$$\begin{aligned} st^2 - u &= 0 \\ s^2 + t - v &= 0, \end{aligned}$$

with Jacobian matrix

$$\begin{pmatrix} t^2 & 2st & -1 & 0 \\ 2s & 1 & 0 & -1 \end{pmatrix}. \tag{3}$$

In general any graph can be written as a level set. We do so such that the Jacobian matrix contains the identity matrix, as in (2), not the negative identity seen in (3)

Vector subspaces in $\mathbb{R}^n$.

7. In the notes it is stated that

i. if $M \in M_{n,r}(\mathbb{R})$ then $\{M\mathbf{t} : \mathbf{t} \in \mathbb{R}^r\}$ is a vector subspace of $\mathbb{R}^n$;

ii. if $N \in M_{m,n}(\mathbb{R})$ then $\{\mathbf{x} \in \mathbb{R}^n : N\mathbf{x} = \mathbf{0}\}$ is a vector subspace of $\mathbb{R}^n$;

iii. if $S \subseteq \mathbb{R}^n$ then the orthogonal complement

$$S^\perp = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \bullet \mathbf{s} = 0 \text{ for all } \mathbf{s} \in S\}$$

is a vector subspace of $\mathbb{R}^n$.

Prove all these assertions.

Solution Throughout let $\alpha, \beta \in \mathbb{R}$.

i. If $\mathbf{u}, \mathbf{v} \in \{M\mathbf{t} : \mathbf{t} \in \mathbb{R}^r\}$ there exist $\mathbf{s}, \mathbf{t} \in \mathbb{R}^r : \mathbf{u} = M\mathbf{s}, \mathbf{v} = M\mathbf{t}$. Then

$$\alpha\mathbf{u} + \beta\mathbf{v} = \alpha M\mathbf{s} + \beta M\mathbf{t} = M(\alpha\mathbf{s} + \beta\mathbf{t}) \in \{M\mathbf{y} : \mathbf{y} \in \mathbb{R}^r\}$$

since $\alpha\mathbf{s} + \beta\mathbf{t} \in \mathbb{R}^r$.

ii. If $\mathbf{u}, \mathbf{v} \in \{\mathbf{x} \in \mathbb{R}^n : N\mathbf{x} = \mathbf{0}\}$ then

$$N(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha N\mathbf{s} + \beta N\mathbf{t} = 0.$$

Hence $\alpha\mathbf{u} + \beta\mathbf{v} \in \{\mathbf{x} \in \mathbb{R}^n : N\mathbf{x} = \mathbf{0}\}$.

iii. If $\mathbf{u}, \mathbf{v} \in S^\perp$ then $\mathbf{u} \bullet \mathbf{s} = 0$ and $\mathbf{v} \bullet \mathbf{s} = 0$ for all $\mathbf{s} \in S$. Thus

$$(\alpha\mathbf{u} + \beta\mathbf{v}) \bullet \mathbf{s} = \alpha\mathbf{u} \bullet \mathbf{s} + \beta\mathbf{v} \bullet \mathbf{s} = 0$$

for all $\mathbf{s} \in S$. Hence $\alpha\mathbf{u} + \beta\mathbf{v} \in S^\perp$.

Note that ii. is a special case of iii; given N let S be the set of rows of N.

Planes in $\mathbb{R}^n$.

8. i. A plane in $\mathbb{R}^3$ is given parametrically by

$$\left\{ \begin{pmatrix} 2x + 4y - 5 \\ 2x + y - 2 \\ 2x - 3y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\}.$$

Express this plane as

a. a graph

$$\left\{ \begin{pmatrix} \mathbf{u} \\ \phi(\mathbf{u}) \end{pmatrix} : \mathbf{u} \in \mathbb{R}^2 \right\},$$

of some function $\phi : \mathbb{R}^2 \to \mathbb{R}$,

b. a level set,

$$f^{-1}(0) = \left\{ \mathbf{s} \in \mathbb{R}^3 : f(\mathbf{s}) = 0 \right\}.$$

for some $f : \mathbb{R}^3 \to \mathbb{R}$.

ii. Repeat for the parametric set

$$\left\{ \begin{pmatrix} 2x + 2y - 2 \\ x + y - 1 \\ 2x - 3y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\}.$$

iii. Repeat for

$$\left\{ \begin{pmatrix} 4x - 4y + 8 \\ -2x + y - 1 \\ 3x - 4y + 6 \\ 4y - 4 \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\},$$

this time expressing this as a graph of some function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, and then as a level set.

Solution i a. In the hope that

$$\left\{ \begin{pmatrix} 2x + 4y - 5 \\ 2x + y - 2 \\ 2x - 3y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\} = \left\{ \begin{pmatrix} \mathbf{u} \\ \phi(\mathbf{u}) \end{pmatrix} : \mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 \right\},$$

write $u = 2x + 4y - 5$ and $v = 2x + y - 2$ and solve:

$$y = \frac{1}{3}(u - v + 3) \quad \text{and} \quad x = \frac{1}{6}(4v - u + 3).$$

Then

$$\phi(\mathbf{u}) = 2x - 3y = \frac{1}{3}(7v - 4u - 6).$$

Thus the given plane is the graph of $\phi(\mathbf{u}) = (7v - 4u - 6)/3$.

b. By the result in part a, as a level set the points $\mathbf{s} = (s, t, u)^T \in \mathbb{R}^3$ on the graph satisfy

$$u = \phi\left((s, t)^T\right), \quad \text{i.e.} \quad 4s - 7t + 3u = -6. \tag{4}$$

So $f(\mathbf{s}) = 4s - 7t + 3u + 6$.

Alternative Approach. Write

$$\begin{pmatrix} 2x+4y-5 \\ 2x+y-2 \\ 2x-3y \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \\ 0 \end{pmatrix} + x \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + y \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}.$$

Then $\mathbf{v}_1 = (2,2,2)^T$ and $\mathbf{v}_2 = (4,1,-3)^T$ span the plane. Thus $\mathbf{v}_1 \wedge \mathbf{v}_2 = (4,-7,3)^T$ is normal to the plane, and for this reason relabeled as $\mathbf{n}$. A definition of the plane is that $\mathbf{s} = (s,t,u)^T$ is in the plane iff $\mathbf{n} \cdot (\mathbf{s} - \mathbf{p}) = 0$ where $\mathbf{p} = (-5,-2,0)^T$. This again leads to (4).

ii. a. There is no hope that

$$\left\{ \begin{pmatrix} 2x+2y-2 \\ x+y-1 \\ 2x-3y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\} = \left\{ \begin{pmatrix} u \\ v \\ \phi(\mathbf{u}) \end{pmatrix} : \mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 \right\},$$

for this would imply

$$u = 2x + 2y - 2 = 2(x + y - 1) = 2v,$$

whereas u and v should be independent variables. Instead we could choose, for example, $u = x + y - 1$ and $v = 2x - 3y$ to get

$$\left\{ \begin{pmatrix} 2x+2y-2 \\ x+y-1 \\ 2x-3y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\} = \left\{ \begin{pmatrix} 2u \\ u \\ v \end{pmatrix} : \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 \right\}.$$

This is a vertical plane in $\mathbb{R}^3$, and it is a graph but where the first coordinate is a function of the second and third coordinates.

b. As a level set this is the set of points $\mathbf{s} = (s,t,u)^T \in \mathbb{R}^3$ such that $s - 2t = 0$; equivalently it is the set $f^{-1}(0)$ for $f(\mathbf{s}) = s - 2t$.

iii. We hope to write the given surface as a graph

$$\left\{ \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\phi}(\mathbf{u}) \end{pmatrix} : \mathbf{s} \in \mathbb{R}^2 \right\} = \left\{ \begin{pmatrix} u \\ v \\ \phi^1(\mathbf{u}) \\ \phi^2(\mathbf{u}) \end{pmatrix} : \mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 \right\},$$

where $\boldsymbol{\phi} : \mathbb{R}^2 \to \mathbb{R}^2$. Hoping that

$$\begin{pmatrix} 4x - 4y + 8 \\ -2x + y - 1 \\ 3x - 4y + 6 \\ 4y - 4 \end{pmatrix} = \begin{matrix} u \\ v \\ \phi^1(\mathbf{u}) \\ \phi^2(\mathbf{u}) \end{matrix}, \tag{5}$$

first solve

$$\begin{aligned} 4x - 4y + 8 &= u \\ -2x + y - 1 &= v. \end{aligned}$$

The solution is

$$x = \frac{-u - 4v + 4}{4} \quad \text{and} \quad y = \frac{-u - 2v + 6}{2}.$$

Then, from (5),

$$\begin{aligned} \phi^1(\mathbf{s}) &= 3x - 4y + 6 = 3\left(\frac{-u - 4v + 4}{4}\right) - 4\left(\frac{-u - 2v + 6}{2}\right) + 6 \\ &= \frac{5u + 4v - 12}{4} \end{aligned}$$

and similarly

$$\phi^2(\mathbf{s}) = 4y - 4 = -2u - 4v + 8.$$

Thus the given plane is the graph of $\boldsymbol{\phi} : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\boldsymbol{\phi}(\mathbf{u}) = \begin{pmatrix} (5u + 4v - 12)/4 \\ -2u - 4v + 8 \end{pmatrix}.$$

The level set is those $(s, t, u, v)^T \in \mathbb{R}^4$ satisfying

$$\begin{aligned} 5s + 4t - 4u &= 12, \\ 2s + 4t + v &= 8. \end{aligned}$$

Equivalently, it is the set $\mathbf{f}^{-1}(\mathbf{0})$ where $\mathbf{f} : \mathbb{R}^4 \to \mathbb{R}^2$,

$$\mathbf{f}(\mathbf{s}) = \begin{pmatrix} 5s + 4t - 4u - 12 \\ 2s + 4t + v - 8 \end{pmatrix},$$

where $\mathbf{s} = (s, t, u, v)^T$.

Level sets are locally graphs

9 i. State the Implicit Function Theorem.

ii. a. Prove, using the Implicit Function Theorem, that for the solutions $(x, y, u, v)^T \in \mathbb{R}^4$ of

$$\begin{aligned} x^2 + y^2 + 2uv &= 4 \\ x^3 + y^3 + u^3 - v^3 &= 0, \end{aligned}$$

there exists an open subset of $\mathbb{R}^4$ containing the solution $\mathbf{p} = (-1, 1, 1, 1)$ in which the u and v can be given as functions of x and y, with $(x, y)^T$ in some open subset of $\mathbb{R}^2$ containing the point $\mathbf{q} = (-1, 1)^T$.

b. Find the partial derivatives of u and v with respect to x and y at $\mathbf{q}$.

c. Is there any open subset of $\mathbb{R}^4$ containing $\mathbf{p}$ in which y and u can be given as functions of x and v? What happens if you attempt to find the partial derivatives of y and u as functions of x and v at this point?

iii. Do the same calculation of partial derivatives for the point $(-1, 1, -1, -1)^T$.

Solution i. Form the notes we have the Implicit Function Theorem: *Suppose that* $\mathbf{f} : U \to \mathbb{R}^m$ *is a* C^1*-function on an open set* $U \subseteq \mathbb{R}^n$ *where* $1 \leq m < n$, *and there exists* $\mathbf{p} \in U$ *such that* $\mathbf{f}(\mathbf{p}) = \mathbf{0}$, $J\mathbf{f}(\mathbf{p})$ *has full-rank* m, *and the final* m *columns of* $J\mathbf{f}(\mathbf{p})$ *are linearly independent. Write*

$$\mathbf{p} = \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix},$$

where $\mathbf{q} \in \mathbb{R}^{n-m}$ *and* $\mathbf{r} \in \mathbb{R}^m$. *Then there exists*

- *an open set* $V : \mathbf{q} \in V \subseteq \mathbb{R}^{n-m}$,
- *a* C^1*-function* $\boldsymbol{\phi} : V \to \mathbb{R}^m$ *and*
- *an open set* $W : \mathbf{p} \in W \subseteq U$

such that for $\left(\mathbf{v}^T, \mathbf{y}^T\right)^T \in W$,

$$\mathbf{f}\left(\begin{pmatrix} \mathbf{v} \\ \mathbf{y} \end{pmatrix}\right) = 0 \textit{ if, and only if } \mathbf{v} \in V \textit{ and} \mathbf{y} = \boldsymbol{\phi}(\mathbf{v}) .$$

ii. a. Define $\mathbf{f} : \mathbb{R}^4 \to \mathbb{R}^2$ by

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x^2 + y^2 + 2uv - 4 \\ x^3 + y^3 + u^3 - v^3 \end{pmatrix}.$$

with $\mathbf{x} = (x, y, u, v)^T$. Then

$$J\mathbf{f}(\mathbf{p}) = \begin{pmatrix} 2x & 2y & 2v & 2u \\ 3x^2 & 3y^2 & 3u^2 & -3v^2 \end{pmatrix}_{\mathbf{x}=\mathbf{p}} = \begin{pmatrix} -2 & 2 & 2 & 2 \\ 3 & 3 & 3 & -3 \end{pmatrix}, \qquad (6)$$

when $\mathbf{p} = (-1, 1, 1, 1)^T$. The last two columns of this matrix $(2, 3)^T$ and $(2, -3)^T$ are linearly independent and so we need no rearrangement of columns to apply the Implicit Function Theorem. In the notation of the Theorem, $\mathbf{q} = (-1, 1)^T$. Thus there exists

- an open set $V : \mathbf{q} \in V \subseteq \mathbb{R}^2$,
- a C^1-function $\boldsymbol{\phi} : V \to \mathbb{R}^2$ and
- an open set $W : \mathbf{p} \in W \subseteq \mathbb{R}^4$

such for $\mathbf{x} \in W$, $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ if, and only if

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in V \quad \text{and} \quad \begin{pmatrix} u \\ v \end{pmatrix} = \boldsymbol{\phi}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

Our required functions are $u = \phi^1$ and $v = \phi^2$, the component functions of vector-valued function $\boldsymbol{\phi}$.

b. Restrict to $(x, y)^T \in V$. Differentiating the equations given in the question with respect to x gives

$$\begin{aligned} 2x + 2\frac{\partial u}{\partial x}v + 2\frac{\partial v}{\partial x}u &= 0 \\ 3x^2 + 3u^2\frac{\partial u}{\partial x} - 3v^2\frac{\partial v}{\partial x} &= 0, \end{aligned}$$

At the point $(x,y)^T = \mathbf{q} = (-1,1)^T$ (when $u = v = 1$) this gives the system

$$\begin{aligned} -2 + 2\frac{\partial u}{\partial x}(\mathbf{q}) + 2\frac{\partial v}{\partial x}(\mathbf{q}) &= 0 \\ 3 + 3\frac{\partial u}{\partial x}(\mathbf{q}) - 3\frac{\partial v}{\partial x}(\mathbf{q}) &= 0, \end{aligned}$$

remembering that u and v, and thus their derivatives, depend only on x and y. Solve this to give

$$\frac{\partial u}{\partial x}(\mathbf{q}) = 0 \quad \text{and} \quad \frac{\partial v}{\partial x}(\mathbf{q}) = 1.$$

Similarly

$$\frac{\partial u}{\partial y}(\mathbf{q}) = -1 \quad \text{and} \quad \frac{\partial v}{\partial y}(\mathbf{q}) = 0.$$

Lesson: *The Implicit Function Theorem says that $u = u(x,y)$ and $v = v(x,y)$ exist though it doesn't say what these functions are. Nonetheless we can find the partial derivatives of these functions.*

c. In the Jacobian matrix in (6) the second and third column, corresponding to y and u are **not** linearly independent. The Implicit Function Theorem does not allow us to conclude that y and u can be given locally as functions of x and v. (**Note** it does **not** imply that y and u can not be given locally as functions of x and v, we just cannot use the Implicit Function Theorem to prove it.)

Aside If you assume that y and u **can** be given locally as functions of x and v, you can attempt to follow the above. Restrict $(x,v)^T$ to some open subset of $\mathbb{R}^2$ containing $\mathbf{q}' = (-1,1)$ (Note that $\mathbf{q}'$ looks identical to $\mathbf{q}$ above, but $\mathbf{q}$ contains the first and second coordinates of $\mathbf{p}$, while $\mathbf{q}'$ the first and fourth.) Differentiating the equations given in the question with respect to x

$$\begin{aligned} 2x + 2y\frac{\partial y}{\partial x} + 2v\frac{\partial u}{\partial x} &= 0 \\ 3x^2 + 3y^2\frac{\partial y}{\partial x} + 3u^2\frac{\partial u}{\partial x} &= 0. \end{aligned}$$

At $\mathbf{q}' = (-1,1)$ (when $x = -1$, $v = 1$) this becomes

$$\begin{aligned} -2 + 2\frac{\partial y}{\partial x}(\mathbf{q}') + 2\frac{\partial u}{\partial x}(\mathbf{q}') &= 0 \\ 3 + 3\frac{\partial y}{\partial x}(\mathbf{q}') + 3\frac{\partial u}{\partial x}(\mathbf{q}') &= 0. \end{aligned}$$

This system is quickly seen to be inconsistent; no solutions exist for $\partial y/\partial x$ and $\partial u/\partial x$.

End of aside

iii. Labelling $\mathbf{p}_2 = (-1, 1, -1, -1)^T$, we have

$$J\mathbf{f}(\mathbf{p}_2) = \begin{pmatrix} -2 & 2 & -2 & -2 \\ 3 & 3 & 3 & -3 \end{pmatrix}.$$

The last two columns are still linearly independent so we can express u and v as functions of x and y in an open set containing $\mathbf{p}_2$. The same method as above gives

$$\frac{\partial u}{\partial x}(\mathbf{q}_2) = -1 \quad \text{and} \quad \frac{\partial v}{\partial x}(\mathbf{q}_2) = 0,$$

and

$$\frac{\partial u}{\partial y}(\mathbf{q}_2) = 0 \quad \text{and} \quad \frac{\partial v}{\partial y}(\mathbf{q}_2) = 1,$$

where $\mathbf{q}_2 = (-1, 1)^T$.

Note that the answers are different to those found in Part ii.b. because the *functions* $u = u(x, y)$ and $v = v(x, y)$ near $\mathbf{p}$ are different to those near $\mathbf{p}_2$. We could have labeled the solutions in Part ii.b. as u_1, v_1 and those from part iii. as u_2, v_2. Then for $(x, y)^T$ close to $(-1, 1)^T$ the points $(x, y, u_1, v_1)^T$ lie close to $\mathbf{p}$ while $(x, y, u_2, v_2)^T$ lie close to $\mathbf{p}_2$.

10. Show that the following level sets are locally graphs around the point given.

i. $(x, y, z)^T \in \mathbb{R}^3 : xy^2z^3 - x^2y^2z^2 + x^3y^2 = 18$ with $\mathbf{p} = (2, 3, -1)^T$.

ii. $(x, y, z)^T \in \mathbb{R}^3$:

$$\begin{aligned} x^2 + 3y^2 + 2z^2 &= 9, \\ xyz &= -2, \end{aligned}$$

with $\mathbf{p} = (2, -1, 1)^T$.

Solution To show that the level set $\mathbf{x} : \mathbf{f}(\mathbf{x}) = \mathbf{0}$ for some C^1-function $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is locally a graph we apply the Implicit Function Theorem. This says that if the final m columns of the Jacobian matrix $J\mathbf{f}(\mathbf{p})$ are linearly independent then the level set is the graph of some function $\boldsymbol{\phi} : V \subseteq \mathbb{R}^{n-m} \to \mathbb{R}^m$.

i. With $f(\mathbf{x}) = xy^2z^3 - x^2y^2z^2 + x^3y^2 - 1$ we have

$$\begin{aligned} Jf(\mathbf{p}) &= f(2,\,3,\,-1) \\ &= \left(y^2z^3 - 2xy^2z^2 + 3x^2y^2,\,2xyz^3 - 2x^2yz^2 + 2x^3y,\,3xy^2z^2 - 2x^2y^2z\right)_{\mathbf{x}=\mathbf{p}} \\ &= (63,\,12,\,126)\,. \end{aligned}$$

The last column is non-zero (the equivalence of linearly independent when only one term). So by the Implicit Function Theorem the last variable, z, can be given as a function of the first two, x and y, in a neighbourhood of $(2,3)^T$.

Note you can also solve for y as a function of x and z in a neighbourhood of $(2,-1)^T$ or for x in terms of y and z in a neighbourhood of $(3,-1)^T$.

ii. The Jacobian of the level set at $\mathbf{p}$ is

$$J\mathbf{f}(\mathbf{p}) = \begin{pmatrix} 2x & 6y & 4z \\ yz & xz & xy \end{pmatrix}_{\mathbf{x}=\mathbf{p}} = \begin{pmatrix} 4 & -6 & 4 \\ -1 & 2 & -2 \end{pmatrix}.$$

The last two columns are linearly independent so the system can be solved with y and z functions of x in a neighbourhood of 2.

11i. Does the equation $x = \sin(xyz)$ determine x as a function of y and z in any open subset of $\mathbb{R}^3$ containing the point $\mathbf{p} = (1,1,\pi/2)^T$, i.e. as a graph $x = \phi(y,z)$?

ii. Does the equation $x = \sin(xyz)$ determine z as a function of x and y in a open subset of $\mathbb{R}^3$ containing the point $\mathbf{p} = (1,1,\pi/2)^T$, i.e. as a graph $z = \phi(x,y)$?

Solution In both parts of this question we are concerned with the level set $f^{-1}(0)$ with $f(\mathbf{x}) = \sin(xyz) - x$ where $\mathbf{x} = (x,y,z)^T$. The Jacobian matrix for f is

$$Jf(\mathbf{x}) = (yz\cos(xyz) - 1, xz\cos(xyz), xy\cos(xyz))\,.$$

Then$Jf(\mathbf{p}) = (-1,0,0)$, full-rank since it is non-zero.

i. The first column in $Jf(\mathbf{p})$ (corresponding to the x variable) is linearly independent (i.e. non-zero). To apply the Implicit Function Theorem as stated in Question 3 we should rearrange the variables so the last column is linearly independent.

Or we just note that, in the notation of the Implicit Function Theorem seen in Question 7, the vector $\mathbf{r}$ in the statement of the Implicit Function Theorem consists of the coordinates of $\mathbf{p}$ corresponding to the independent columns. Here there is only one such column so r is a scalar, in fact $r = 1$. The vector $\mathbf{q}$ contains all other coordinates of $\mathbf{p}$, so $\mathbf{q} = (1, \pi/2)^T$.

The conclusion is that there exists an open set $V : \mathbf{q} \in V \subseteq \mathbb{R}^2$, a C^1-function $\phi : V \to \mathbb{R}$ and an open set $W : \mathbf{p} \in W$ such for $(x, y, z)^T \in W$, $\sin(xyz) - x = 0$ if, and only if $(y, z)^T \in V$ and $x = \phi\left((y, z)^T\right)$.

Hence x can be given as a function of y and z in some open subset of $\mathbb{R}^3$ containing the point $\mathbf{p} = (1, 1, \pi/2)^T$.

ii. Since the last column in $Jf(\mathbf{p}) = (-1, 0, 0)$ is linearly **dependent** (because it is zero), the Implicit Function Theorem tells us nothing; z may be a function of x and y or it may not.

We have to approach the problem differently. For example, we can attempt to solve $\sin(xyz) - x = 0$ giving $z = \left(\sin^{-1} x\right)/xy$. Yet in any *open* set containing $(1, 1, \pi/2)^T$ there will be points $(x, y, z)^T$ with $x > 1$. But $\sin^{-1} x$ is not defined for such x hence z cannot be given as a function of x and y in any open set containing $(1, 1, \pi/2)^T$.

Solutions to Additional Questions 6

12. Prove part of a Theorem from the Notes: $P \subseteq \mathbb{R}^n$ is a plane of dimension r iff

i. there exists a point $\mathbf{p} \in \mathbb{R}^n$ and a full rank matrix $M \in M_{n,r}(\mathbb{R})$ such that $P = \{\mathbf{p} + M\mathbf{t} : \mathbf{t} \in \mathbb{R}^r\}$,

ii. there exists a point $\mathbf{p} \in \mathbb{R}^n$ and a full rank matrix $N \in M_{n-r,n}(\mathbb{R})$ such that $P = \{\mathbf{x} \in \mathbb{R}^n : N(\mathbf{x} - \mathbf{p}) = \mathbf{0}\}$.

Hint for part ii. If $\mathcal{V} \subseteq \mathbb{R}^n$ is a vector space then $\dim \mathcal{V}^\perp = n - \dim \mathcal{V}$. (For a proof see appendix of Section 3 Part 1.)

The important part of these results is the relationship between the dimension of the plane and the fact that the matrices are of full rank

Solution i. By Question 7 $\{M\mathbf{t} : \mathbf{t} \in \mathbb{R}^r\}$ is a vector space and as stated in the notes

$$\{M\mathbf{t} : \mathbf{t} \in \mathbb{R}^r\} = \operatorname{span}\{\mathbf{c}_1, ..., \mathbf{c}_r\} \tag{7}$$

where the $\mathbf{c}_i$ are the columns of M. Then

$$\begin{aligned} M \text{ is of full rank} \quad & \text{iff} \quad \dim \operatorname{span}\{\mathbf{c}_1, ..., \mathbf{c}_r\} = r \\ & \text{iff} \quad \dim\{M\mathbf{t} : \mathbf{t} \in \mathbb{R}^r\} = r \quad \text{by (7)} \\ & \text{iff} \quad \{\mathbf{p} + M\mathbf{t} : \mathbf{t} \in \mathbb{R}^r\} \quad \text{is a plane of dimension } r. \end{aligned}$$

ii. By Question 7 $\{\mathbf{x} \in \mathbb{R}^n : N\mathbf{x} = \mathbf{0}\}$, is a vector space and as stated in the notes

$$\{\mathbf{x} \in \mathbb{R}^n : N\mathbf{x} = \mathbf{0}\} = \operatorname{span}\{\mathbf{r}_1, ..., \mathbf{r}_{n-r}\}^\perp \tag{8}$$

where the $\mathbf{r}_i$ are the rows of N. Then

$$\begin{aligned} N \text{ is of full rank} \quad & \text{iff} \quad \dim \operatorname{span}\{\mathbf{r}_1, ..., \mathbf{r}_{n-r}\} = n - r \\ & \text{iff} \quad \dim \operatorname{span}\{\mathbf{r}_1, ..., \mathbf{r}_{n-r}\}^\perp = n - (n - r) \quad \text{by hint} \\ & \text{iff} \quad \dim\{\mathbf{x} \in \mathbb{R}^n : N\mathbf{x} = \mathbf{0}\} = r \quad \text{by (8)} \\ & \text{iff} \quad \{\mathbf{x} \in \mathbb{R}^n : N(\mathbf{x} - \mathbf{p}) = \mathbf{0}\} = \mathbf{p} + \{\mathbf{y} \in \mathbb{R}^n : N\mathbf{y} = \mathbf{0}\} \end{aligned}$$

is a plane of dimension r.

13 Let $\phi(\mathbf{u}) = 2u^2 + 3uv - 4v^2$ for $\mathbf{u} = (u, v)^T \in \mathbb{R}^2$. Then $\mathbf{p} = (1, 2, -8)^T$ is a point on the graph of ϕ. In which direction $\mathbf{v} \in \mathbb{R}^2$ is the fastest ascent? the fastest descent? no change in height?

Solution $\mathbf{q} = (1,2)^T$ and $\nabla\phi(\mathbf{q}) = (4u+3v, 3u-8v)^T_{\mathbf{u}=\mathbf{q}} = (10,-13)^T$. So the direction of greatest ascent is $(10,-13)^T/\sqrt{269}$, of greatest descent $(-10,13)^T/\sqrt{269}$.

There will be no change in height when $d_{\mathbf{v}}\phi(\mathbf{q}) = 0$, i.e. $\phi(\mathbf{q}) \bullet \mathbf{v} = 0$. This is $\mathbf{v} = \pm(13,10)^T/\sqrt{269}$.

14. Define the function

$$f(\mathbf{x}) = \frac{x^2y + 2xy^2}{1 + x^2 + y^2} \quad \text{for} \quad \mathbf{x} = (x,y)^T \in \mathbb{R}^2.$$

Imagine standing on the graph of f above the point $\mathbf{q} = (1,2)^T$ and spilling water. In which direction would the water flow?

Solution Note first that $f(\mathbf{q}) = 5/3$. Next, multiply up

$$\left(1 + x^2 + y^2\right) f(\mathbf{x}) = x^2y + 2xy^2$$

Taking the partial derivatives w.r.t x,

$$2xf(\mathbf{x}) + \left(1 + x^2 + y^2\right)\frac{\partial f}{\partial x}(\mathbf{x}) = 2xy + 2y^2,$$

so

$$\frac{10}{3} + 6\frac{\partial f}{\partial x}(\mathbf{q}) = 4 + 8, \quad \text{i.e} \quad \frac{\partial f}{\partial x}(\mathbf{q}) = \frac{13}{9}.$$

Similarly, taking the partial derivatives w.r.t y,

$$2yf(\mathbf{x}) + \left(1 + x^2 + y^2\right)\frac{\partial f}{\partial y}(\mathbf{x}) = x^2 + 4xy$$

so

$$\frac{20}{3} + 6\frac{\partial f}{\partial y}(\mathbf{q}) = 9 \quad \text{i.e} \quad \frac{\partial f}{\partial y}(\mathbf{q}) = \frac{7}{18}.$$

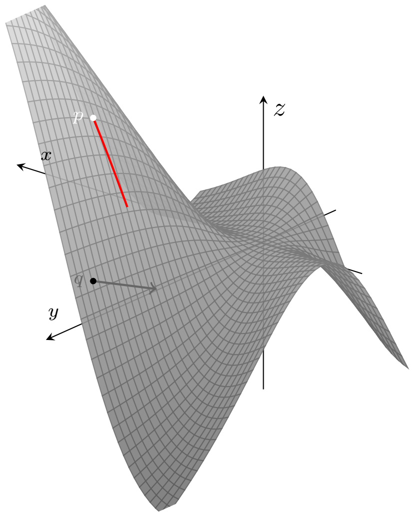
Thus

$$\nabla f(\mathbf{q}) = \frac{1}{18}\begin{pmatrix} 26 \\ 7 \end{pmatrix}.$$

Then presumably the water will run down the path of fastest descent, which is in the direction of

$$-\frac{\nabla f(\mathbf{q})}{|\nabla f(\mathbf{q})|} = -\frac{1}{\sqrt{725}}\begin{pmatrix} 26 \\ 7 \end{pmatrix}.$$

The graph $z = f(\mathbf{x})$ for Question 14:



As we look at the surface the point $\mathbf{q}$ lies in the x - y plane underneath the surface.

15. Let $T : \mathbb{R}^3 \to \mathbb{R}, x \mapsto 10 - 2e^{2x^2+3y^2+z^2}$ give the temperature at each point in $\mathbb{R}^3$.

i. In which direction from $\mathbf{p} = (2, 0, 2)^T$ does the temperature increases as quickly as possible? Decreases as quickly as possible?

ii. Let $S \subseteq \mathbb{R}^3$ be a surface in $\mathbb{R}^3$ given parametrically as

$$\left\{ \begin{pmatrix} u^2 + v \\ u - v \\ uv + u \end{pmatrix} : 0 \le u, v \le 2 \right\}.$$

The point $\mathbf{p} = (2, 0, 2)^T \in S$ is the image of $\mathbf{q} = (1, 1)^T$. If a spider stands at $\mathbf{p}$, and is restricted to stay **on** the surface, in which direction must they move to increase the temperature as quickly as possible; to decrease it as quickly as possible?

Solution i. The gradient vector of T at $\mathbf{p}$ is

$$\nabla T(\mathbf{x}) = -e^{2x^2+3y^2+z^2} \begin{pmatrix} 8x \\ 12y \\ 4z \end{pmatrix} \quad \text{so} \quad \nabla T(\mathbf{p}) = -e^{-12} \begin{pmatrix} 10 \\ 0 \\ 8 \end{pmatrix}.$$

Look back at Question 9 on Sheet 5 to see that greatest increase in temperature is in direction

$$\frac{\nabla T(\mathbf{p})}{|\nabla T(\mathbf{p})|} = -\frac{1}{\sqrt{41}}\begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}.$$

The greatest decrease is in the opposite direction (multiply by -1).

ii . Let

$$\mathbf{F}(\mathbf{u}) = \begin{pmatrix} u^2+v \\ u-v \\ uv+u \end{pmatrix},$$

for $\mathbf{u} = (u,v)^T : 0 \leq u, v \leq 2$. Then the temperature restricted to the surface is $T(\mathbf{F}(\mathbf{u})) = T \circ \mathbf{F}(\mathbf{u})$. So the greatest increase in the temperature at $\mathbf{q}$ is in the direction of $\nabla(T \circ \mathbf{F})(\mathbf{q})$ (and the greatest decrease in $-\nabla(T \circ \mathbf{F})(\mathbf{q})$).

Recall $\nabla(T \circ \mathbf{F})(\mathbf{q}) = J(T \circ \mathbf{F})(\mathbf{q})^T$ while the Chain Rule gives

$$J(T \circ \mathbf{F})(\mathbf{q}) = JT(\mathbf{F}(\mathbf{q}))\, J\mathbf{F}(\mathbf{q}).$$

These combine as

$$\nabla(T \circ \mathbf{F})(\mathbf{q}) = J\mathbf{F}(\mathbf{q})^T\, \nabla T(\mathbf{F}(\mathbf{q})) = J\mathbf{F}(\mathbf{q})^T\, \nabla T(\mathbf{p}).$$

Yet

$$J\mathbf{F}(\mathbf{q}) = \begin{pmatrix} 2u & 1 \\ 1 & -1 \\ v+1 & u \end{pmatrix}_{\mathbf{u}=\mathbf{q}} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 2 & 1 \end{pmatrix}.$$

Thus

$$\nabla(T \circ \mathbf{F})(\mathbf{q}) = -e^{-12}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 10 \\ 0 \\ 8 \end{pmatrix} = -e^{-12}\begin{pmatrix} 18 \\ 36 \end{pmatrix}.$$

We only need the direction, so moving from $\mathbf{q}$ in the direction of $-(1,2)^T/\sqrt{5}$ the temperature on the surface will increase at the greatest rate, in the direction $(1,2)^T/\sqrt{5}$ it decreases at the greatest rate.

To compare this result with that in Part i. we need to know what, as we move from $\mathbf{q}$ in direction $\mathbf{v}$, is the direction from $\mathbf{p}$ in $\mathbb{R}^3$? As we move from $\mathbf{q}$ in direction $\mathbf{v}$ we move on the surface along the curve $\mathbf{F}(\mathbf{q}+t\mathbf{v})$.

The direction of travel at $t = 0$ is the derivative which, by the Chain Rule, is $J\mathbf{F}(\mathbf{q})\,\mathbf{v}$. Thus the fastest increase is in the direction

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \text{i.e.} \quad \frac{1}{\sqrt{42}} \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix}.$$

Again, the fastest decrease is in the opposite direction.

16. i. Prove that

$$\begin{aligned} xe^y + uz - \cos(v\pi/2) &= 2 \\ u\cos(y\pi/2) + x^2v - yz^2 &= 1, \end{aligned}$$

can be solved for u, v in terms of x, y, z near $\mathbf{p} = (2, 0, 1, 1, 0)^T$. (The general point of $\mathbb{R}^5$ is $(x, y, z, u, v)^T$).

ii. Can you find a point $\mathbf{p}'$ around which the system can be solved for x and z in terms of y, u and v?

Solution If we set

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} xe^y + uz - \cos(v\pi/2) - 2 \\ u\cos(y\pi/2) + x^2v - yz^2 - 1 \end{pmatrix}$$

then

$$J\mathbf{f}(\mathbf{x}) = \begin{pmatrix} e^y & xe^y & u & z & \pi\sin(v\pi/2)/2 \\ 2xv & -u\pi\sin(y\pi/2)/2 - z^2 & -2yz & \cos(y\pi/2) & x^2 \end{pmatrix}.$$

Thus

$$J\mathbf{f}(\mathbf{p}) = \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 4 \end{pmatrix}.$$

Since the fourth and fifth columns (corresponding to the variables u and v) are linearly dependent the Implicit Function Theorem says that in some set $W \subseteq \mathbb{R}^5 : \mathbf{p} \in W$ a point $\mathbf{x} \in W$ satisfies $\mathbf{f}(\mathbf{x}) = 0$ iff $(x, y, x)^T \in V$ for some open set V and $(u, v)^T = \boldsymbol{\phi}(x, y, x)$ for some C^1-function $\boldsymbol{\phi} : V \to \mathbb{R}^2$.

ii. First note that the point $\mathbf{p}$ will not suffice; the columns corresponding to x and z, the first and third are not linearly independent (remember, this does not say that it is not solvable for x and z in terms of y, u and v, but rather we cannot prove it is).

So we have to find another solution to our system. We can simplify some exponentials and trig. functions by keeping $y = 0$ but this time choose $v = 1$. The system then becomes

$$x + uz = 2 \quad \text{and} \quad u + x^2 = 1.$$

This has many solutions so we look at the Jacobian matrix with $y = 0$ and $v = 1$:

$$\begin{pmatrix} 1 & x & u & z & \pi/2 \\ 2x & -z^2 & 0 & 1 & x^2 \end{pmatrix}.$$

For the first and third columns to be linearly independent we only require $x \neq 0$. A possible solution would then be $x = 1/2, u = 3/4$ and $z = 2$. Hence choose $\mathbf{p}' = (2, 0, 2, 3/4, 2)^T$ when

$$J\mathbf{f}(\mathbf{p}') = \begin{pmatrix} 1 & 2 & 3/4 & 2 & \pi/2 \\ 1 & -4 & 0 & 1 & 1/4 \end{pmatrix}.$$

In fact, in this matrix every pair of columns are linearly independent so, in some neighbourhood of $\mathbf{p}'$, the system can be solved for any two variables in terms of the other three.

17. Can $(x^2 + y^2 + 2z^2)^{1/2} = \cos z$ be solved for y in terms of x and z near $(0, 1, 0)^T$?

Solution Writing $f(\mathbf{x}) = (x^2 + y^2 + 2z^2)^{1/2} - \cos z$ for $\mathbf{x} \in \mathbb{R}^3$ the Jacobian matrix is

$$Jf(\mathbf{x}) = \frac{1}{(x^2 + y^2 + 2z^2)^{1/2}} \left(x, y, 2z + (x^2 + y^2 + 2z^2)^{1/2} \sin z \right).$$

At $\mathbf{p} = (0, 1, 0)^T$ this becomes

$$Jf(\mathbf{p}) = (0, 1, 0).$$

Since the second column corresponding to the variable y is non-zero (equivalent to the notation of linearly independent when the column 'vectors' have only one component) the Implicit Function Theorem says that the system can be solved near p for y as a function of x and z.